ON THE RELATION BETWEEN CAUCHY'S NUMBERS AND BESSEL'S FUNCTIONS.

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It is sometimes of advantage in Celestial Mechanics to transform formulas expressed by means of Cauchy's numbers into a form where Bessel's functions are the instrument of calculations and *vice versa*. It is therefore useful to know what relations there exist between Cauchy's numbers and Bessel's functions.

In a note on Cauchy's numbers by the author* it has been shown that

$$N_{-p,j,q} = \left[\frac{j}{n}\right] - \left[\frac{j}{n-1}\right]\left[\frac{q}{1}\right] + \left[\frac{j}{n-2}\right]\left[\frac{q}{2}\right] - \ldots + (-1)^n \left[\frac{q}{n}\right] \tag{1}$$

where $n = \frac{1}{2}(-p + j + q)$, and the symbol $\left[\frac{a}{b}\right]$ stands for

$$\frac{a(a-1)\ldots(a-b+1)}{b!}.$$

This formula gives for $j = 0$

$$N_{-p,0,q} = (-1)^n \left[\frac{q}{n}\right] = (-1)^n \frac{q!}{n!\,(q-n)!} = (-1)^n \frac{q!}{n!\,(n+p)!}. \tag{2}$$

On the other hand we have for Bessel's function $J_i(x)$ the expression

$$J_i(x) = \sum_{m=0}^{m=\infty} (-1)^m \frac{\left[\frac{x}{2}\right]^{i+2m}}{m!\,(i+m)!}. \tag{3}$$

Therefore we can write

$$J_i(x) = \sum_{m=0}^{m=\infty} \frac{N_{-i,0,i+2m}}{(i+2m)!}\left[\frac{x}{2}\right]^{i+2m} \tag{4}$$

and, differentiating with regard to x,

$$\frac{dJ_i(x)}{dx} = \frac{1}{2}\sum_{m=0}^{m=\infty} \frac{N_{-i,0,i+2m}}{(i+2m-1)!}\left[\frac{x}{2}\right]^{i+2m-1}. \tag{5}$$

As an application of these formulas some properties of Bessel's functions

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* Annals of Mathematics, Vol. 10, p. 1.

can be derived from the properties of Cauchy's numbers. Thus, for example, by (4)

$$J_{-i}(x) = \sum_{m=0}^{m=\infty}{}' \frac{N_{i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m} \tag{6}$$

and, on account of

$$N_{-p,j,q} = (-1)^p N_{p,j,q}$$

formula (6) becomes

$$J_{-i}(x) = (-1)^i \sum_{m=0}^{m=\infty}{}' \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m},$$

or, since $N_{-p,j,q} = 0$ for $-p+j+q<0$, and therefore $N_{-i,0,-i+2m} = 0$ for $m<i$,

$$J_{-i}(x) = (-1)^i \sum_{m=i}^{m=\infty}{}' \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[\frac{x}{2}\right]^{-i+2m} = (-1)^i \sum_{m=0}^{m=\infty}{}' \frac{N_{-i,0,2m+i}}{(2m+i)!} \left[\frac{x}{2}\right]^{2m+i},$$

that is to say

$$J_{-i}(x) = (-1)^i J_i(x).$$

In a similar way we can derive the well known relations

$$iJ_i(x) = \frac{x}{2}\left[J_{i-1}(x) + J_{i+1}(x)\right],$$

$$\frac{dJ_i(x)}{dx} = \tfrac{1}{2}\left[J_{i-1}(x) - J_{i+1}(x)\right],$$

from the following properties of Cauchy's numbers:

$$N_{-p,j+1,q} = N_{-p+1,j,q} + N_{-p-1,j,q}$$

$$N_{-p,j,q+1} = N_{-p+1,j,q} - N_{-p-1,j,q}.$$

As another application of our formulas let us express the function $\left[\frac{r}{a}-1\right]^m$ of Celestial Mechanics by means of Bessel's functions given the formulas*

$$\left[\frac{r}{a}-1\right]^m = \tfrac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty}{}' c_i^{(m)} \cos i\zeta$$

$$c_i^{(m)} = (-1)^m \frac{2m}{i}\left[\frac{e}{2}\right]^m \sum_{k=0}^{k=\infty}{}' \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \tag{7}$$

* See *Tisserand*, Mécanique Céleste, tome I, pp. 237–239. For the values of $c_0^{(m)}$ see l. c. formulas (20) and (21). These values are also given below, namely, by formulas (9) and (10).

To this end we remark first that the number $N_{-p,j,q}$ can be expressed linearly in Cauchy's numbers of the form $N_{-p',0,q}$ by the formula*

$$N_{-p,j,q} = N_{-p+j,0,q} + \left[\frac{j}{1}\right] N_{-p+j-2,0,q} + \left[\frac{j}{2}\right] N_{-p+j-4,0,q} + \ldots + N_{-p-j,0,q}\,,$$

which becomes in the present case

$$\begin{aligned} N_{-i,m-1,i-m+1+2k} = {} & N_{-i+m-1,0,i-m+1+2k} \\ & + \left[\frac{m-1}{1}\right] N_{-i+m-3,0,i-m+1+2k} \\ & + \left[\frac{m-1}{2}\right] N_{-i+m-5,0,i-m+1+2k} \\ & + \ldots\ldots\ldots\ldots \\ & + N_{-i-m+1,0,i-m+1+2k}\,. \end{aligned}$$

It must be noticed that in the first line of this formula $k \geqq 0$, in the second line $k \geqq 1$, in the third $k \geqq 2$, and so on; in the last line $k \geqq (m-1)$ because $N_{-p,j,q} = 0$ when $-p+j+q$ is negative. Hence

$$\begin{aligned} \sum_{k=0}^{k=\infty}{}' \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} = {} & \sum_{k=0}^{k=\infty}{}' \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{1}\right]\sum_{k=1}^{k=\infty}{}' \frac{N_{-i+m-3,0,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{2}\right]\sum_{k=2}^{k=\infty}{}' \frac{N_{-i+m-5,0,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \sum_{k=m-1}^{k=\infty}{}' \frac{N_{-i-m+1,0,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \\ = {} & \sum_{k=0}^{k=\infty}{}' \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!}\left[\frac{ie}{2}\right]^{i-m+2k} \\ & + \left[\frac{m-1}{1}\right]\sum_{k=0}^{k=\infty}{}' \frac{N_{-i+m-3,0,i-m+3+2k}}{(i-m+2k+2)!}\left[\frac{ie}{2}\right]^{i-m+2k+2} \end{aligned}$$

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* Annals of Mathematics, Vol. 10; l. c.

$$+ \left[\frac{m-1}{2}\right] \sum_{k=0}^{k=\infty} \frac{N_{-i+m-5,\,0,\,i-m+5+2k}}{(i-m+2k+4)!} \left[\frac{ie}{2}\right]^{i-m+2k+4}$$

$$+ \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot$$

$$+ \sum_{k=0}^{k=\infty} \frac{N_{-i-m+1,\,0,\,i+m+2k-1}}{(i+m+2k-2)!} \left[\frac{ie}{2}\right]^{i+m+2k-2}$$

$$= \frac{2}{i}\frac{d}{de}$$

$$\left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \left[\frac{m-1}{2}\right] J_{i-m+5}(ie) + \ldots + J_{i+m-1}(ie) \right\}$$

and therefore

$$c_i^{(m)} = (-1)^m \frac{4m}{i^2} \left[\frac{e}{2}\right]^m \frac{d}{de}$$

$$\left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \left[\frac{m-1}{2}\right] J_{i-m+5}(ie) + \ldots + J_{i+m-1}(ie) \right\}$$

which is the formula we intended to derive.

Conversely, suppose the function $\left[\frac{r}{a}-1\right]^m$ expressed with the help of Bessel's functions. To obtain this expression directly, we remark that

$$\left[\frac{r}{a}-1\right]^m = (-1)^m \left[\frac{e}{2}\right]^m 2\,.\,(2^{m-1}\cos^m u)$$

$$= (-1)^m \left[\frac{e}{2}\right]^m$$

$$2\left\{ \cos mu + \left[\frac{m}{1}\right] \cos(m-2)u + \left[\frac{m}{2}\right] \cos(m-4)u + \ldots \right\}$$

the last term within the brackets being $\frac{1}{2}\left[\frac{2m_1}{m_1}\right]$ if $m = 2m_1$; and $\left[\frac{2m_1+1}{m_1}\right]$ $\cos u$ if $m = 2m_1 + 1$. In either case the above development will contain a constant term which we will denote by $\frac{1}{2}\, c_0^{(m)}$. When $m = 2m_1$

$$\tfrac{1}{2}\, c_0^{(m)} = (-1)^m \left[\frac{e}{2}\right]^m \left[\frac{2m_1}{m_1}\right] = \frac{(m_1+1)(m_1+2)\ldots 2m_1}{m_1!} \left[\frac{e}{2}\right]^{2m_1} \tag{9}$$

when $m = 2m_1 + 1$, since $\cos u = -\frac{1}{2}e +$ periodic terms, we shall have the constant term

$$\tfrac{1}{2}\, c_0^{(m)} = -\tfrac{1}{2}\, e(-1)^m \left[\frac{e}{2}\right]^m 2 \left[\frac{2m_1+1}{m_1}\right] = 2\,\frac{(m_1+2)(m_1+3)\ldots(2m_1+1)}{m_1!} \left[\frac{e}{2}\right]^{m+1}. \tag{10}$$

Formulas (9) and (10) are identical with (20) and (21) of Tisserand (l. c. p. 238).

We have on the other hand for $m > 1$

$$\cos mu = m \sum_{i=1}^{i=\infty} [J_{i-m}(ie) - J_{i+m}(ie)] \frac{\cos i\zeta}{i},$$

which formula holds also for $m = 1$ if we leave out the non-periodic term. Hence

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + (-1)^m \left[\frac{e}{2}\right]^m 2 \sum_{i=1}^{i=\infty} A_i \frac{\cos i\zeta}{i}, \tag{11}$$

where we have put

$$A_i = m \left\{ J_{i-m}(ie) - J_{i+m}(ie) \right\} + (m-2) \left[\frac{m}{1}\right] \left\{ J_{i-m+2}(ie) - J_{i+m-2}(ie) \right\}$$
$$+ (m-4) \left[\frac{m}{2}\right] \left\{ J_{i-m+4}(ie) - J_{i+m-4}(ie) \right\} + \dots.$$

Now, it can be readily shown that

$$(m-2p) \left[\frac{m}{p}\right] = m \left\{ \left[\frac{m-1}{p}\right] - \left[\frac{m-1}{p-1}\right] \right\}$$

so that

$$A_i = m \left\{ [J_{i-m}(ie) - J_{i-m+2}(ie)] + \left[\frac{m-1}{1}\right] [J_{i-m+2}(ie) - J_{i-m+4}(ie)] + \dots \right\}$$
$$= \frac{2m}{i} \frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \dots \right\},$$

and, therefore,

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + (-1)^m 4m \left[\frac{e}{2}\right]^m \sum_{i=1}^{i=\infty} \frac{\cos i\zeta}{i^2} \frac{d}{de}$$
$$\left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \dots \right\}$$

We have shown above that

$$\frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[\frac{m-1}{1}\right] J_{i-m+3}(ie) + \dots \right\}$$
$$= \frac{i}{2} \sum_{k=0}^{k=\infty} \frac{N_{-i,\, m-1,\, i-m+1+2k}}{(i-m+2k)!} \left[\frac{ie}{2}\right]^{i-m+2k}$$

Hence, $c_i^{(m)}$ being given by formula (7) we shall have

$$\left[\frac{r}{a} - 1\right]^m = \tfrac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty} c_i^{(m)} \cos i\zeta$$

which solves the proposed problem.

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